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It is well known that solutions of the backwards heat equation can be approximated by solutions of a pseudoparabolic equation depending on a small parameter $\varepsilon$. The emphasis in this paper is on the mathematical problems which arise in approximating solutions to initial-boundary value problems for this perturbed equation. The approximation procedure we propose is obtained through the development of a potential theory for pseudoparabolic equations, the asymptotic evaluation of certain contour integrals, and results based on a theorem of Levin in the theory of entire functions.

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# THE APPROXIMATION OF SOLUTIONS TO THE BACKWARDS HEAT EQUATION BY SOLUTIONS OF PSEUDOPARABOLIC EQUATIONS

by

DAVID COLTON*

Summary

It is well known that solutions of the backwards heat equation can be approximated by solutions of a pseudoparabolic equation depending on a small parameter $\varepsilon$. The emphasis in this paper is on the mathematical problems which arise in approximating solutions to initial-boundary value problems for this perturbed equation. The approximation procedure we propose is obtained through the development of a potential theory for pseudoparabolic equations, the asymptotic evaluation of certain contour integrals, and results based on a theorem of Levin in the theory of entire functions.

## I. Introduction.

The problem of constructing solutions to the heat equation backwards in time is one of the classical improperly posed problems in partial differential equations. Mathematically the problem can be formulated (in $R^3$) in the following manner: For D a bounded domain in $R^3$ find $u(\underset{\sim}{x},t)$, $\underset{\sim}{x} \in R^3$, such that

$$\Delta_3 u = u_t \text{ in } D \times (0,t_o) \tag{1.1a}$$

$$u = 0 \text{ on } \partial D \times (0,t_o) \tag{1.1b}$$

$$u(\underset{\sim}{x},t_o) = \phi(\underset{\sim}{x}) \text{ in } D \tag{1.1c}$$

where $\phi(\underset{\sim}{x})$ is a prescribed function. As is well known ([15], [16]) in general no solution exists to (1.1a) - (1.1c) and if it does the solution does not depend continuously on the data $\phi(\underset{\sim}{x})$ in any reasonable norm. One approach to "solving" (1.1a) - (1.1c) is the method of quasi-reversibility as

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initially developed by Lattes and Lions ([11]), and it is a version of this approach that we wish to discuss in this paper. The version we have in mind is to replace the problem (1.1a) - (1.1c) by the modified problem

$$\varepsilon \Delta_3 u_t - u_t + \Delta_3 u = 0 \text{ in } D \times (0,t_o) \tag{1.2a}$$

$$u = 0 \text{ on } \partial D \times (o,t_o) \tag{1.2b}$$

$$u(\underset{\sim}{x},t_o) = \phi(\underset{\sim}{x}) \text{ in } D \tag{1.2c}$$

where $\varepsilon$ is a small positive parameter ([7], [9]). Three obvious questions immediately present themselves:

1) Does a unique solution to (1.2a) - (1.2c) exist for every $\varepsilon > 0$ and does it depend continuously on the data $\phi(\underset{\sim}{x})$?
2) As $\varepsilon \to 0$ does the solution of (1.2a) - (1.2c) approach the solution of (1.1a) - (1.1c) (if it exists!)?
3) What constructive methods are available for approximating the solution of (1.2a) - (1.2c) for $\varepsilon > 0$?

In most discussions of the method of quasi-reversibility attention is usually focused on questions 1) and 2), and the answer to 3) is normally "use finite difference approximations or partial eigenfunction expansions". However for multi-dimensional problems, large time intervals, and small values of the parameter $\varepsilon$, such methods are in general rather impractical. It is in fact with this difficulty in mind that we have chosen to use the third order perturbed equation (1.2a) instead of the fourth order equation

$$\varepsilon \Delta_3^2 u - u_t + \Delta_3 u = 0 \tag{1.3}$$

as proposed by Lattes and Lions ([11]). As will be seen, such a choice will enable us to use potential theoretic methods and asymptotic analysis to examine solutions of (1.2a) - (1.2c) for small $\varepsilon$, thus providing a more practical method for constructing approximations to solutions of the backwards heat equation in higher dimensions and for large time intervals.

Having tailored our model (1.2a) - (1.2c) to give a satisfactory answer to question 3) does not of course excuse us from dealing with the first two questions! However, the question of existence, uniqueness, and continuous dependence on the data is well known (c.f. [1]) and the second question has been answered by Ewing in the following theorem ([7]):

<u>Theorem 1</u>: Let $u(\underset{\sim}{x},t)$ be a solution of (1.1a), (1.1b) such that $||u(\underset{\sim}{x},t_o) - \phi(\underset{\sim}{x})||_{L^2} < \delta$, $||u(\underset{\sim}{x},0)||_{L^2} < M$, where $\delta, M$ are positive constants, and let $v(\underset{\sim}{x},t)$ be the solution of (1.2a) - (1.2c) for $\varepsilon = [\log {}^M/_\delta]^{-1}$. Then for every $t > 0$,

$$||u-v||_{L^2} \leq C(t)\varepsilon$$

where $C(t)$ does not depend on $\varepsilon$.

An obvious drawback of Ewing's result is that the error bound depends logarithmically on $\delta$ instead of Hölder continuously as in the stabilized quasi-reversibility method of Millar ([15]). A similiar problem also arises if one uses quasi-reversibility methods in conjunction with (1.3), or, in a different direction, Tikhonov's regularization method for solving the backwards heat equation (c.f. [8], [18]). In our case this means that in order to achieve accurate results we must choose the parameter $\varepsilon$ to be quite small and assume that the data at $t = t_o$ is measured with a high degree of accuracy.

Although our analysis in what follows is presented for the heat equation in $R^3$, analogous results can also be obtained in $R^2$ (see Section IV).

## II. <u>Potential Theory for the Pseudo-Heat Equation</u>.

Equation (1.2a) is a particular example of an equation of pseudo-parabolic or Sobolev type and is usually referred to as the pseudo-heat equation. Equations of pseudoparabolic type appear in a variety of areas of application and have been the object of a considerable amount of attention in recent years. For information concerning this class of equations as well as an extensive bibiography we suggest consulting the recent book by Carroll and Showalter ([1]). In this section we shall develop a potential theory for the pseudo-heat equation ([3], [5]) with the aim of using these results in

Section III to construct approximations to the solution of (1.2a) - (1.2c). We shall develop this potential theory for the "forward" initial-boundary value problem

$$\varepsilon\,\Delta_3 u_t - u_t + \Delta_3 u = 0 \text{ in } D\times(0,t_o) \qquad (2.1a)$$

$$u = f(\underset{\sim}{x},t) \text{ on } \partial D \times (0,t_o) \qquad (2.1b)$$

$$u(\underset{\sim}{x},0) = 0 \text{ in } D \qquad (2.1c)$$

where $f(\underset{\sim}{x},t)$ is assumed to be continuously differentiable and reserve until Section III the modifications and applications of these results to the study of (1.2a) - (1.2c).

We begin by defining the fundamental solution for (2.1a) by ([3], [5])

$$\Gamma(R,\, t-\tau) = -\frac{1}{\pi i R} \oint_{|\omega - \frac{1}{\sqrt{\varepsilon}}| = \delta} \exp\left[-\omega R + \frac{\omega^2 (t-\tau)}{1-\varepsilon\omega^2}\right] d\omega \qquad (2.2)$$

where $R = |\underset{\sim}{x} - \underset{\sim}{\xi}|$ for $\underset{\sim}{x},\, \underset{\sim}{\xi} \in R^3$ and the path of integration is a circle of radius $\delta$ traversed counterclockwise about the point $\omega = \frac{1}{\sqrt{\varepsilon}}$. If we now assume that D is a bounded, simply connected domain with Lyapunov boundary $\partial D$ and let $\rho(\underset{\sim}{\xi},\tau)$ denote a continuous density defined on $\partial D \times [0,t_o]$, we can define a pseudo-heat potential by

$$u(\underset{\sim}{x},t) = \frac{1}{2\pi}\int_0^t \int_{\partial D} \rho(\underset{\sim}{\xi},\tau)\, \frac{\partial^2}{\partial\nu\partial\tau}\, \Gamma(R,\, t-\tau)\, ds\, d\tau \qquad (2.3)$$

where $\nu$ denotes the unit normal on $\partial D$ pointing into D. In order for (2.3) to be a solution of (2.1a) - (2.1c) it is necessary to choose $\rho(\underset{\sim}{\xi},\tau)$ such that (2.1b) is satisfied. To this end we differentiate both sides of (2.3) with respect to $t$ and let $\underset{\sim}{x} \to \partial D$. Then from the discontinuity properties of metaharmonic potentials we arrive at an integral equation for $\rho(\underset{\sim}{\xi},\tau)$ of the form

$$\varepsilon^{3/2}\,\frac{\partial f}{\partial t} = (\underset{\sim}{I} + \underset{\sim}{T} + \underset{\sim}{L}_1 + \underset{\sim}{L}_2)\rho \qquad (2.4)$$

where $\underset{\sim}{T}$ is a Fredholm integral operator over $\partial D$, $\underset{\sim}{L}_1$ is a Volterra integral operator over $[0,t]$, and $\underset{\sim}{L}_2$ is a Volterra integral operator over $\partial D \times [0,t]$. The kernels of these integral operators depend on $\Gamma(R,t)$ and its derivatives (c.f.[3]). Due to the fact that $\varepsilon > 0$ we can deduce the following Theorem ([3]):

Theorem 2: Let $C(\partial D \times [0,t_o])$ be the Banach space of continuous functions defined over $\partial D \times [0,t_o]$ with respect to the maximum norm. Then $(\underset{\sim}{I} + \underset{\sim}{T} + \underset{\sim}{L}_1 + \underset{\sim}{L}_2)^{-1}$ exists as a bounded linear operator on $C(\partial D \times [0,t_o])$. Hence any solution of (2.1a) - (2.1c) can be represented in the form (2.3) for some continuous density $\rho(\underset{\sim}{\xi},\tau)$.

In order to actually construct $\rho(\underset{\sim}{\xi},\tau)$ from (2.4) it is necessary to determine accurate approximations to $\Gamma(R,t)$ and its derivatives for small values of $\varepsilon$ and $R \geq 0$, $t \geq 0$. Clearly it suffices to consider the function

$$\begin{aligned} K(R,t) &= -\frac{1}{\pi i} \oint_{|\omega - \frac{1}{\sqrt{\varepsilon}}| = \delta} \exp\left[-\omega R + \frac{\omega^2 t}{1-\varepsilon\omega^2}\right] d\omega \\ &= -\frac{\exp(-t/\varepsilon)}{\sqrt{\varepsilon}\,\pi i} \oint_{|\mu - 1| = \delta} \exp\left[-\mu \frac{R}{\sqrt{\varepsilon}} + \frac{t}{\varepsilon(1-\mu^2)}\right] d\mu . \end{aligned} \tag{2.5}$$

The evaluation of $K(R,t)$ can be conveniently divided into three separate cases ([5]):

1) R=0, t=0(1). In this case we expand $\exp\left(\frac{t}{\varepsilon(1-\mu^2)}\right)$ in its Taylor series in $t$ and integrate termwise to arrive at

$$K(0,t) = \frac{t \exp(-{}^{t}/_{\varepsilon})}{\varepsilon}\, \Phi({}^{1}/_{2}, 2; {}^{t}/_{\varepsilon}) \tag{2.6}$$

where $\Phi(a,c;z)$ denotes the confluent hypergeometric function. From the asymptotic behaviour of $\Phi(a,c;z)$ for large $z$ (c.f.[6]) we have from (2.6) that

$$K(0,t) = \frac{1}{\sqrt{\pi t}} \left[ \sum_{n=0}^{N} \frac{({}^1/_2)_n ({}^3/_2)_n}{n!} \left(\frac{\varepsilon}{t}\right)^n + 0\left(\frac{\varepsilon}{t}\right)^{N+1} \right] \qquad (2.7)$$

$$\text{where } (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} .$$

2) <u>$R\geq 0$, $t=0(\varepsilon)$</u>. We again expand $\exp\left(\frac{t}{\varepsilon(1-\mu^2)}\right)$ in powers of t and integrate termwise to obtain

$$K(R,t) = \frac{2\exp(-{}^t/_\varepsilon)}{\sqrt{\varepsilon\pi}} \sum_{n=0}^{\infty} \frac{1}{(n+1)!\,n!} \left(\frac{t}{\varepsilon}\right)^{n+1} \left(\frac{R}{2\sqrt{\varepsilon}}\right)^{n+{}^1/_2} K_{n+{}^1/_2}\left(\frac{R}{\sqrt{\varepsilon}}\right) \qquad (2,8)$$

where $K_n(z)$ denotes the modified Bessel function. For $t = 0(\varepsilon)$, (2.8) can be readily approximated by truncating the series and applying the Clenshaw-Luke method of backward recursion ([13], Section 11.8). An estimate on the convergence ratio of (2.8) can be obtained by using the inequality

$$\left|\left(\frac{R}{2\sqrt{\varepsilon}}\right)^{n+{}^1/_2} K_{n+{}^1/_2}\left(\frac{R}{\sqrt{\varepsilon}}\right)\right| \leq {}^1/_2 \;\; \Gamma(n+{}^1/_2) \qquad (2.9)$$

3) <u>$R>0$, $t=0(1)$</u>. By deforming the circle $\left|\omega - \frac{1}{\sqrt{\varepsilon}}\right| = \delta$ in (2.5) onto the contour pictured below in Figure 1

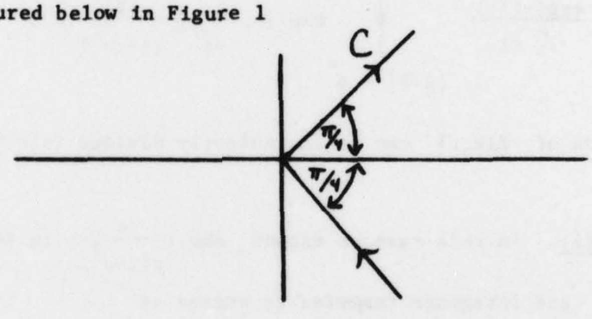


Figure 1

we can rewrite $K(R,t)$ in the form

$$K(R,t) = \frac{1}{\pi i\sqrt{t}} \int_C \exp\left[-\frac{R}{\sqrt{t}} z + z^2\right] g_\varepsilon(z)dz \qquad (2.10)$$

where

$$g_\varepsilon(z) = \exp\left[\frac{\varepsilon z^4}{t-\varepsilon z^2}\right]. \qquad (2.11)$$

By using ad hoc methods, it is now possible to obtain a complete asymptotic expansion for $K(R,t)$ as $\varepsilon \to 0$ in the form

$$K(R,t) = \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{R^2}{4t}\right)\left[1+d_1\left(\frac{\varepsilon}{t}\right)+d_2\left(\frac{\varepsilon}{t}\right)^2+\ldots+d_n\left(\frac{\varepsilon}{t}\right)^n+O\left(\left(\frac{\varepsilon}{t}\right)^{n+1}\right)\right] \qquad (2.12)$$

where the coefficients $d_j$ are expressible in terms of Hermite polynomials ([5]). In particular

$$d_1 = \frac{1}{4^2} H_4\left(\frac{R}{2\sqrt{t}}\right)$$

$$d_2 = \frac{1}{2\cdot 4^4} H_8\left(\frac{R}{2\sqrt{t}}\right) + \frac{1}{4^3} H_6\left(\frac{R}{2\sqrt{t}}\right) \qquad (2.13)$$

with similar expressions holding for the higher order coefficients. In (2.13) $H_n(z)$ denotes Hermite's polynomial.

The expansions in all of the above cases may be differentiated termwise.

## III. Approximation of Solutions to the "Backwards" Pseudo-Heat Equation.

We now discuss the problem of approximating solutions of initial-boundary value problems for the "backwards" pseudo-heat equation defined by (1.2a) - (1.2c). One approach proceeds as follows. By replacing $t$ by $t_o-t$, and using the Fourier transform or Borel transform to construct a solution to the pure initial value problem, we can reduce problem (1.2a) - (1.2c) to an initial-boundary value problem of the form

$$\varepsilon\, \Delta_3 u_t - u_t - \Delta_3 u = 0 \text{ in } D \times (0,t_o) \qquad (3.1a)$$

$$u = f(\underset{\sim}{x},t) \text{ on } \partial D \times (0,t_o) \qquad (3.1b)$$

$$u(\underset{\sim}{x},0) = 0 \text{ in } D. \qquad (3.1c)$$

Note that (3.1a) - (3.1c) differs from (2.1a) - (2.1c) only in a sign change in the differential equation. Although this has no effect on the well-posedness of the problem, as we shall see it unfortunately has serious implications on the problem of constructing approximate solutions for small values of the (positive) parameter $\varepsilon$. This is, of course, not surprising for the limiting case as $\varepsilon \to 0$ is now an improperly posed problem, i.e. the backwards heat equation. If we follow the analysis of the previous section, it is seen that we can represent the solution of (3.1a) - (3.1c) in the form

$$u(\underset{\sim}{x},t) = \frac{1}{2\pi}\int_0^t \int_{\partial D} \rho(\underset{\sim}{\xi},\tau)\frac{\partial^2}{\partial\nu\partial\tau}\Gamma(R,\tau-t)ds d\tau \qquad (3.2)$$

where $\rho(\underset{\sim}{\xi},\tau)$ is determined as the solution of an integral equation of Fredholm-Volterra type and the notation is the same as in (2.3). We observe that the only difference between (3.2) and (3.3) is that the argument of the fundamental solution is $\tau-t$ instead of $t-\tau$. This change, however, unfortunately complicates the evaluation of $\Gamma(R,t)$ since we are now interested in the asymptotic behaviour of $\Gamma(R,t)$ for $t < 0$ instead of $t > 0$. Except for the expansion (2.8) the analysis of Section II breaks down and we are forced into adopting an ad hoc approach yielding expansions of $\Gamma(R,t)$ in terms of a series of generalized hypergeometric functions or a Bessel-Laguerre series (c.f.[4]). Such expansions are of limited value in the evaluation of $\Gamma(R,t)$ for $t = O(1)$ and hence we are led to look for other approximation procedures.

To present such an alternate approach we return to the formulation (1.2a) - (1.2c). We first note that if $\lambda j$ is a sequence

$$\lim_{j\to\infty} \frac{j}{\lambda_j^2} > 0$$

then in a sufficiently small complex neighbourhood of $[0,t_o]$ we can approximate any analytic function defined in this neighbourhood by a finite linear combination of functions taken from the set $\{e^{-\lambda_j^2 t}\}_{j=1}^{\infty}$. This result follows from a theorem of Levin in the theory of entire functions and the reader is referred to [12], p. 219, for details. From the results of [2] we can now conclude that any continuous function defined in $\overline{D}$ can be approximated in $\overline{D}$ by a linear combination of the functions

$$r^{-1/2} J_{n+1/2} (\lambda_j r) P_n^m(\cos \Theta) e^{im\phi}; \quad n=0,1,2,\ldots \; ; -n\le m\le n \qquad (3.4)$$

where $J_n(z)$ denotes Bessel's function and $P_n^m(z)$ an associated Legendre function. We observe that

$$u_{nmj} (\underset{\sim}{x},t) = r^{-1/2} J_{n+1/2} (\lambda_j r) P_n^m(\cos \Theta) \exp \left[im\phi - \frac{\lambda_j^2}{1 + \varepsilon\lambda_j^2} t\right] \qquad (3.5)$$

is a solution of (1.2a). Although the set $\{u_{nmj}\}$ is a complete set of solutions when $\varepsilon=0$([2]), for $\varepsilon>0$ the order of the equation is increased and hence except for exceptional circumstances we would expect that extra functions would have to be introduced to supplement the above set. (An exceptional case occurs for example when D is a sphere and $\{\lambda_j^2\}$ are the eigenvalues of the Laplacian in D.) In particular from the Runge approximation property for pseudoparabolic equations ([17]) and the results of Section II (approximating the pseudo-heat potential defined on the cylinder $S \times [0,t_o] \supset D \times [0,t_o]$ by numerical quadrature) we have the following Theorem, where completeness is with respect to the $L^2$ norm over $D \times [0,t_o]$.

<u>Theorem 3</u>: Let $\{\lambda_j\}$ be as in (3.3) and let $\{\underset{\sim}{\xi}_j\}_{j=1}^{\infty}$ be a dense set of points on a sphere $S \supset D$. Then the functions

$$w_{nj}(\underset{\sim}{x},t) = \int_0^t \tau^n \frac{\partial^2}{\partial\nu\partial\tau} \Gamma(|\underset{\sim}{\xi}_j - \underset{\sim}{x}|, t-\tau)d\tau$$

$$u_{nmj} (\underset{\sim}{x},t) = r^{-1/2} J_{n+1/2} (\lambda_j r) P_n^m(\cos \Theta) \exp \left[im\phi - \frac{\lambda_j^2}{1+\varepsilon\lambda_j^2} t\right]$$

where $\nu = \underset{\sim}{\xi}_j / |\underset{\sim}{\xi}_j|$ form a complete set of solutions to (1.2a) defined in $D \times [0,t_o]$.

Now let $\{\Psi_n\}_{n=1}^{\infty}$ denote the complete set of functions defined in Theorem 3. In order to find an approximate solution to (1.2a) - (1.2c) we minimize the functional

$$||\sum_{n=1}^{N} c_n\Psi_n(\underset{\sim}{x},t_o) - \phi(\underset{\sim}{x})||_{L^2(D)} + ||\sum_{n=1}^{N} c_n\Psi_n(\underset{\sim}{x},t)||_{L^2(\partial D \times [0,t_o])} \tag{3.6}$$

and define the approximate solution to be $\sum_{n=1}^{N} c_n\Psi_n(\underset{\sim}{x},t)$. The functions $\Psi_n$ can be evaluated by using existing tables for the computation of Bessel and Legendre functions in combination with the results of Section II to approximate $\frac{\partial^2}{\partial\nu\partial\tau}\Gamma\ |\underset{\sim}{\xi} - x|,\ t-\tau)$. Due to the fact that (1.2a) - (1.2c) is well posed, the minimization procedure indicated above is stable. Following Gaponenko ([10]) we require that the minimization of (3.6) allow not only for the variation of the coefficients $c_n$ but also for the basis functions $\Psi_n$ themselves, i.e. by considering $\underset{\sim}{\xi}_j$, $j = 1, \ldots, N$, and $\lambda_j$, $j = 1, \ldots, N$, to be variables as well. Thus we are led to the problem of finding the minimum of a function of the 6 N variables $\mathrm{Re}\ c_n$, $\mathrm{Im}\ c_n$, $\underset{\sim}{\xi}_j = (\xi_j^{(1)}, \xi_j^{(2)}, \xi_j^{(3)})$, $\lambda_j$, where $1 \le j \le N$. A related minimization problem is considered in [10], together with its numerical implementation by the conjugate gradient method, and a similar approach for elliptic boundary value problems has also been discussed by Mathon ([14]).

## IV. The Backwards Heat Equation in $R^2$.

The analysis of the previous sections can be extended to treat the backwards heat equation in $R^2$, and in this section we briefly outline the necessary modifications. Theorem 1 is valid in any number of dimensions. A potential theory for the two dimensional pseudo-heat equation can be developed exactly as in Section II where the fundamental solution is now defined by

$$\Gamma(R,t-\tau) = -\frac{1}{\pi i}\oint_{|\omega - \frac{1}{\sqrt{\varepsilon}}| = \delta} K_o(\omega R)\exp\left[\frac{\omega^2(t-\tau)}{1-\varepsilon\omega^2}\right] d\omega \tag{4.1}$$

with $K_o(z)$ denoting a modified Bessel function of order zero. From [6] we have the representation

$$K_o(z) = \int_1^\infty e^{-zs} (s^2-1)^{-1/2}\, ds \qquad (4.2)$$

and hence from (2.5) we can write

$$\Gamma(R,t) = \int_1^\infty K(Rs,t)(s^2 - 1)^{-1/2}\, ds. \qquad (4.3)$$

Expansion formulae for $\Gamma(R,t)$ can now be obtained for $R>0$, $t=0(\varepsilon)$ from (2.8) and for $R>0$, $t=0(1)$ from (2.12) since these expansions can be integrated termwise with respect to R over the range $(1,\infty)$. In particular from (2.12) we have

$$\Gamma(R,t) = \frac{1}{\sqrt{\pi t}} \int_1^\infty \exp\left(-\frac{R^2 s^2}{4t}\right) (s^2 - 1)^{-1/2}\, ds + 0\left(\frac{\varepsilon}{t}\right)$$

$$= \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{R^2}{8t}\right) \int_1^\infty \exp\left(-\frac{R^2 \mu}{8t}\right) (\mu^2 - 1)^{-1/2}\, d\mu + 0\left(\frac{\varepsilon}{t}\right) \qquad (4.4)$$

$$= \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{R^2}{8t}\right) K_o\left(\frac{R^2}{8t}\right) + 0\left(\frac{\varepsilon}{t}\right).$$

The coefficients of the higher order terms can be expressed in terms of Whittaker functions. Finally, the analysis of Section III proceeds in the same manner except that the functions $r^{-1/2} J_{n+1/2} (\lambda_j r)\, P_n^m(\cos \Theta) e^{im\phi}$ are now $J_n(\lambda_j r) e^{\pm in\Theta}$.

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